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Long-period oscillations in a harbour induced by incident short waves

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Progressive short waves with a narrow frequency band are known to be accompanied by long set-down waves travelling with the groups. In finite depth, scattering of short waves by a large structure or a varying coastline can radiate free long waves which propagate faster than the incident set-down. In a partially enclosed harbour attacked by short waves through the entrance, such free long waves can further resonate the natural modes of the harbour basin. In this paper an asymptotic theory is presented for a harbour whose horizontal dimensions are much greater than the entrance width, which is in turn much wider than the short wavelength.

1. Introduction

Excessive oscillations inside a harbour can be detrimental to the mooring and docking of ships and the loading or unloading of cargoes. For many harbours, the most important natural modes have rather long periods, several minutes to an hour. Most studies in the past have been focused on a linear mechanism of resonant scattering due to incident long waves such as tsunamis. Since the pioneering work of Miles & Munk (1961) the linear resonance theory has been well developed; effective numerical techniques now exist and are used in practice (see Mei 1983 for a survey). Extensions to account for nonlinear effects which help to distribute energy to higher harmonics (Rogers & Mei 1978; Lepelletier 1980), boundary friction and flow separation at the entrance which augments dissipation (Ito 1970; Ünlüata & Mei 1975), have also received some attention.

Only a few of the major ports in the world are threatened by tsunamis, however, while all of them are under regular assault from storm induced waves of much shorter periods ($O(10\text{ s})$). Significant oscillations of moored ships at the period of several minutes are often reported not only in harbours along sea coasts (Santas Lopez & Gomez Pina 1988) but also along the shores of the Great Lakes of North America. It is therefore very important to develop an effective theory for the prediction of such occurrences.

Because of the sharp difference in frequencies, wind waves are ineffective for exciting oscillations in a harbour directly by the linear mechanism of resonant scattering. Based on field observations it was suggested long ago that nonlinearity may transfer energy from groups of short waves to long surf beats (Munk 1949), which can in turn excite harbour oscillations. Longuet-Higgins & Stewart (1962) have found theoretically that the radiation stress, due to nonlinear convective inertia, can generate long-period set-down waves which follow the envelope of progressive short waves with a narrow frequency band. Bowers (1977) was the first to relate this set-down wave to harbour oscillations. Specifically, he treated a semi-infinite long channel which has a narrower bay at its end. Slowly modulated incident waves arriving towards the narrower bay induce not only set-down waves locked to the incident and scattered wave envelopes, but also free long waves which propagate at the faster speed of the shallow water waves $(gh)^{\frac{1}{2}}$. It is these free waves which are resonated in the bay under certain conditions. He chose the range of wavelengths so that the propagation was one-dimensional both in and outside the bay. Most harbours are, however, situated near an open coast, so that the corresponding problem involves both two-dimensional diffraction and nonlinearity. Because of algebraic complexity, direct perturbation analysis for a general frequency spectrum is not easy. The difficulty is amply illustrated by past controversies on the simplest problem of nonlinear diffraction by a vertical circular cylinder (see e.g. references cited in Mei 1983).

In recent years a number of nonlinear diffraction and refraction problems have been studied for narrow-banded short waves by using in part the method of multiple scales. For example Agnon & Mei (1985), Agnon, Choi & Mei (1988) have treated the slow drift of two-dimensional floating cylinders, while Zhou & Liu (1987) extended the analysis to the diffraction by a vertical circular cylinder. Agnon & Mei (1988*a*) also examined the resonance of long trapped waves on a submarine ridge by short waves. In a study of the diffraction of surface waves on a thin vertical barrier in a two-layered sea, Agnon & Mei (1988*b*) found that long-period internal waves can be excited in the shadow of the short surface waves. In this paper we shall show that similar reasoning can be used to give simple insight into and prediction for the nonlinear resonance of long waves in a harbour by groups of short waves. For analytical convenience attention will be limited to a sea of constant depth and a straight coast. The harbour entrance is assumed to be much wider than the short wavelength but much smaller than the largest dimension of the harbour basin as well as the long wavelength. Detailed analysis will be given for a narrow bay of rectangular plan form. Short waves are dealt with by the geometric optics method complemented by the parabolic approximation. Because the short waves change rapidly across certain rays, the forced long waves are discontinuous there. A simple free long wave is then added to ensure continuity. A second free wave is combined with the first to satisfy the no-flux condition along the entire coastline without the harbour entrance. Finally, a third free wave is constructed to account for the harbour. The dependence of long-wave resonance on the short waves is then discussed. Extensions to more complex harbour shapes are made at the end.

2. Estimation of scales

For a simple progressive wave of slope $\epsilon = ka$, it is convenient and sufficiently general to assume that the slow modulation rate which is related to the narrow width of the frequency band $\Delta\omega/\omega$, is also of the same order $O(\epsilon)$, although in nature these two small parameters are physically unrelated. Then the wave groups have the lengthscales and timescales $K^{-1} = O(\epsilon k)^{-1}$ and $\Omega^{-1} = O(\epsilon\omega)^{-1}$ respectively. In the linearized theory of habours, it is well known that the lowest modes including the Helmholtz mode, with $KL \leqq O(1)$ where L is the typical harbour width or length, are amplified the most. The amplification factor depends not only on KL but also on the entrance width W which controls the radiation damping. In particular, the amplification factor at resonance is of the order δ^{-1} where $\delta = KW$ for a narrow bay and $\delta = (\log KW)^{-1}$ for all but the Helmholtz modes in a harbour with comparable dimensions in all horizontal directions. In practice, the typical entrance width is $W = O(100\text{–}1000)$ m, so that kW is rather large compared to unity where k is the wavenumber of the wind waves. Thus

$$KW \ll 1 \ll kW, \tag{2.1}$$

is quite commonly encountered in reality. Moreover, it is quite common that

$$1 \geqslant O(\delta) \geqslant O(\epsilon^{\frac{1}{2}}). \tag{2.2}$$

To have some idea of the corresponding physical dimensions we consider two narrow bays with $h = 20$ m and assume $\delta = \epsilon^{\frac{1}{2}}$, $K = \epsilon k$, $\epsilon = 0.01$; then

T (period, s)	kh	k (m^{-1})	W (m)	L (km)	KL	KW
18	0.5	0.025	400	5	1.25	0.1
9	1.2	0.06	167	1	1	0.1

The width and length in the first and second rows are common for shipping and fishing harbours respectively. However, for analytical insight, we begin with a more restrictive assumption that

$$1 > O(\delta) > O(\epsilon^{\frac{1}{2}}), \tag{2.3}$$

(for example, $\delta = O(\epsilon^{\frac{1}{3}})$). As will be shown, the long-wave equations so obtained have a broader range of applicability up to $\delta = O(1)$, therefore hold under the conditions specified by (2.2). But the approximate solution to be developed is based on (2.3). The case of $\delta = O(1)$ for which resonance is not expected, would require some numerical work and will not be discussed here.

3. Approximate governing equations

For explaining the reasoning it suffices to consider the simple example of normal incidence on a narrow bay of length L and width $2W$ as sketched in figure 1. Assuming irrotational flow in an inviscid fluid, we seek a velocity potential Φ that satisfies Laplace's equation

$$\nabla^2\Phi = 0 \quad (-h < z < \zeta), \tag{3.1}$$

where ζ denotes the free-surface displacement. The bottom depth h is taken to be constant and all sidewalls are vertical and impervious; thus,

$$\frac{\partial \phi}{\partial x} = 0 \quad \begin{cases} x = 0, & \text{if } |y| > W, \\ x = -L, & \text{if } |y| < W, \end{cases} \tag{3.2}$$

and

$$\frac{\partial \phi}{\partial y} = 0 \quad (0 > x > -L, \quad |y| = W). \tag{3.3}$$

By Taylor expansion about $z = 0$ the kinematic and dynamic boundary conditions on the free surface are:

$$\zeta_t + \Phi_x \zeta_x + \Phi_y \zeta_y = \Phi_z + \zeta\Phi_{zz} + \dots \quad (z = 0), \tag{3.4}$$

$$g\zeta + \Phi_t + \zeta\Phi_{tz} + \tfrac{1}{2}(\nabla\Phi)^2 = 0 + \dots \quad (z = 0). \tag{3.5}$$

where terms cubic in amplitude are omitted. These two conditions can be combined to give

$$\Phi_{tt} + g\Phi_z = \frac{\partial}{\partial t}\left[-\tfrac{1}{2}(\nabla\Phi)^2 + \frac{1}{g}\Phi_t\Phi_{tz}\right] - \nabla_h \cdot (\Phi_t \nabla_h \Phi) + \dots \quad (z = 0), \tag{3.6}$$

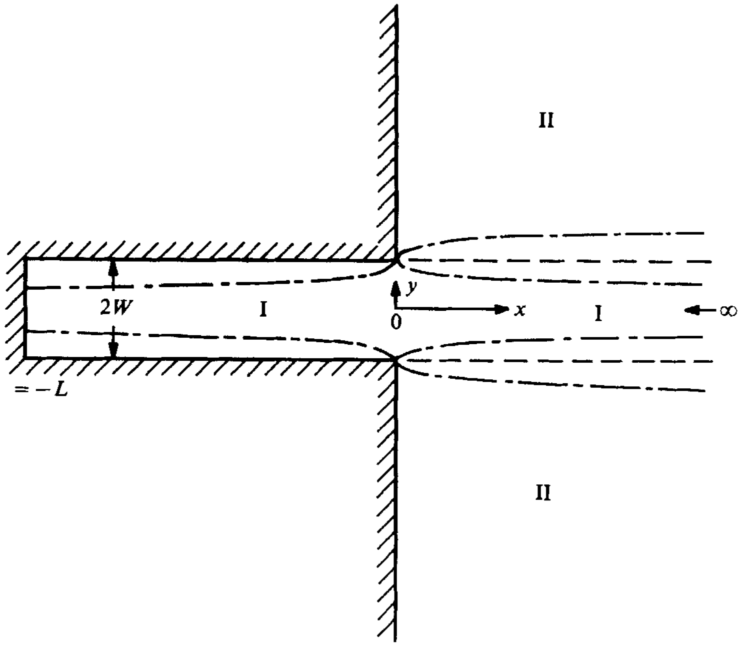

FIGURE 1. Definitions for a rectangular bay. Dashed curves indicate the edges of the parabolic region of diffraction.

with
$$\nabla_h = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right). \tag{3.7}$$

As is known, a simple progressive wavetrain is always accompanied by a second-order long wave (set-down) propagating at the group velocity. Owing to the diffraction of short waves we further expect free long waves which propagate at the shallow water speed $(gh)^{\frac{1}{2}}$ and can be resonated in the bay to the order $O(\epsilon^2/\delta)$. For thin breakwaters or entrances with sharp corners, diffraction of short waves is confined in parabolic boundary layers whose width is such that $ky = O(kx)^{\frac{1}{2}}$, as sketched in figure 1. For the region exterior to these boundary layers we introduce the multiple-scale variables:
$$x_1 = \epsilon x, \quad y_1 = \epsilon y, \quad t_1 = \epsilon t. \tag{3.8}$$

These coordinates are clearly appropriate for the outer regions outside the bay. Within the bay, there are also two half parabolic layers. The outer region is of width $ky = O(\delta/\epsilon)$ which is $O(\epsilon^{-\frac{2}{3}})$ if $\delta = O(\epsilon^{\frac{1}{3}})$, and will be treated in terms of y_1 with the understanding that $1 \gg O(ky_1) = O(\delta) \gg O(\epsilon^{\frac{1}{2}})$.

The perturbation expansions:
$$\Phi = \epsilon\left(\frac{1}{\delta}\phi_1^{(-1)} + \phi_1^{(0)} + \delta\phi_1^{(1)} + \dots\right) + \epsilon^2\left(\frac{1}{\delta}\phi_2^{(-1)} + \phi_2^{(0)} + \delta\phi_2^{(1)} + \dots\right), \tag{3.9}$$

and
$$\zeta = \epsilon\zeta_1 + \epsilon^2\left(\frac{1}{\delta}\zeta_2^{(-1)} + \zeta_2^{(0)} + \delta\zeta_2^{(1)} + \dots\right), \tag{3.10}$$

are assumed for the outer regions, where
$$\phi_n^{(-1)} = \phi_n^{(-1)}(z; x_1, y_1, t_1), \quad \phi_n^{(m)} = \phi_n^{(m)}(x, y, z, t; x_1, y_1, t_1), \tag{3.11}$$
$$\zeta_n^{(-1)} = \zeta_n^{(-1)}(x_1, y_1, t_1), \quad \zeta_n^{(m)} = \zeta_n^{(m)}(x, y, t; x_1, y_1, t_1), \tag{3.12}$$

for $m = 0, 1, 2, \ldots$. The rationale of these expansions is that at the first order in ϵ, the short-wave displacement ($\epsilon\zeta_1$) is not resonated. The long-wave displacement which is of the second order in ϵ is resonated with the peak amplitude $O(\epsilon^2/\delta)$, hence it is represented to leading order by $\epsilon^2\zeta_2^{(-1)}/\delta$. Correspondingly, to the leading order, the short-wave potential is described by $\epsilon\phi_1^0$ while the resonated long-wave potential is described by $(\epsilon/\delta)\phi_1^{-1}$. Note that the spatial or time derivatives of $(\epsilon/\delta)\phi_1^{(-1)}$ are of the order $O(\epsilon^2/\delta)$ and are directly related to $\epsilon^2\zeta_2^{(-1)}/\delta$. At each order the solution can be further decomposed into harmonics with respect to the short-wave frequency ω. The perturbation procedure is very similar to those applied in our previous papers and will only be sketched here.

4. The short wave

At the leading order $O(\epsilon)$, ζ_1 is dominated by the short waves, while $\phi_1^{(0)}$ consists of both short and long waves. The perturbation equations are those familiar in the linearized theory:

$$\nabla^2\phi_1^{(0)} = 0, \tag{4.1}$$

$$\frac{\partial^2\phi_1^{(0)}}{\partial t^2} + g\phi_1^{(0)} = 0 \quad (z = 0), \tag{4.2}$$

$$g\zeta_1 + \frac{\partial\phi_1^{(0)}}{\partial t} = 0 \quad (z = 0), \tag{4.3}$$

and

$$\frac{\partial\phi_1^{(0)}}{\partial n} = 0, \tag{4.4}$$

along all solid walls. This potential may be written

$$\phi_1^{(0)} = \phi_{10} + (\phi_{11}\,\mathrm{e}^{-\mathrm{i}\omega t} + *), \tag{4.5}$$

where ϕ_{11} satisfies the Helmholtz equation in the horizontal plane and the radiation condition at infinity. The zeroth harmonic ϕ_{10} corresponds to a part of the long wave and can be shown to depend only on the slow coordinates, as in, for example, Agnon & Mei (1985).

Since $kW \gg 1$, the geometrical optics (ray) and the parabolic approximations can be combined here to achieve an analytical solution. Referring to figure 1 we divide the fluid domain into regions I and II by the edge rays $x \geqslant 0$, $y = \pm W$. According to the ray approximation, we have

Region I:

$$\zeta_{11} = \tfrac{1}{2}(a_-\mathrm{e}^{-\mathrm{i}kx} + a_+^{\mathrm{I}}\,\mathrm{e}^{\mathrm{i}k(x+2L)}), \tag{4.6}$$

$$\phi_{11} = -\frac{\mathrm{i}g\cosh k(z+h)}{2\omega\cosh kh}(a_-\,\mathrm{e}^{-\mathrm{i}kx} + a_+^{\mathrm{I}}\,\mathrm{e}^{\mathrm{i}k(x+2L)}), \tag{4.7}$$

Region II:

$$\zeta_{11} = \tfrac{1}{2}(a_-\,\mathrm{e}^{-\mathrm{i}kx} + a_+^{\mathrm{II}}\,\mathrm{e}^{\mathrm{i}kx}), \tag{4.8}$$

$$\phi_{11} = -\frac{\mathrm{i}g\cosh k(z+h)}{2\omega\cosh kh}(a_-\,\mathrm{e}^{-\mathrm{i}kx} + a_+^{\mathrm{II}}\,\mathrm{e}^{\mathrm{i}kx}). \tag{4.9}$$

The amplitudes a_-, a_+^{I} and a_+^{II} of the incident and reflected waves are slowly modulated and depend on x_1 and t_1 according to the law of wave-action conservation. From the no-flux condition at the order $O(\epsilon^2)$,

$$\frac{\partial\phi_{11}}{\partial x_1} = 0 \quad \text{on walls along } x_1 = 0, \quad x_1 = -L_1, \tag{4.10}$$

where $L_1 \equiv \epsilon L$, therefore

$$a_- = a\left(t_1 + \frac{x_1}{C_g}\right), \quad a_+^{\mathrm{I}} = a\left(t_1 - \frac{(x_1+2L_1)}{C_g}\right), \quad a_+^{\mathrm{II}} = a\left(t_1 - \frac{x_1}{C_g}\right). \tag{4.11}$$

Except for the special cases where $2kL = n\pi$ and a is periodic with the period $2L_1/mC_g$, m = integer, the reflected short wave ζ_{11} is discontinuous across the edge rays ($x \geqslant 0$, $y = \pm W$). This discontinuity can be remedied by inserting the transition factor $D(x, y)$ which accounts for diffraction according to the parabolic approximation. The following result is uniformly valid for all x and y except near the corners of the bay entrance

$$\zeta_{11} = \tfrac{1}{2}a_- \,\mathrm{e}^{-\mathrm{i}kx} + \tfrac{1}{2}a_+^{\mathrm{I}}\,\mathrm{e}^{\mathrm{i}k(x+2L)} + \tfrac{1}{2}(a_+^{\mathrm{II}} - a_+^{\mathrm{I}}\,\mathrm{e}^{2\mathrm{i}kL})\,D\,\mathrm{e}^{\mathrm{i}kx}, \tag{4.12}$$

where

$$D = D^- + D^+, \tag{4.13a}$$

$$D^{\pm} = \frac{1}{\sqrt{2}}\mathrm{e}^{-\frac{1}{4}\mathrm{i}\pi}\left[\tfrac{1}{2} + C(\beta^{\pm}) + \mathrm{i}(\tfrac{1}{2} + S(\beta^{\pm}))\right], \tag{4.13b}$$

$$\beta^{\pm} = \frac{k(\pm y - W)}{(\pi k x)^{\frac{1}{2}}}. \tag{4.13c}$$

The superscripts $(\cdot)^{\pm}$ refer to the rays $y = \pm W$. C and S are the Fresnel cosine and sine integrals respectively. A similar expression for ϕ_{11} is omitted. While the details of D are unimportant for later purposes, we note that diffraction is confined in two parabolic boundary layers defined by

$$k(\pm y - W) = O((\pi k x)^{\frac{1}{2}}). \tag{4.14}$$

Thus for $kx_1 = \epsilon kx = O(1)$ the width of these boundary layers is $O(ky) = O(\epsilon^{-\frac{1}{2}})$.

5. The long wave and bay resonance

First we consider the geometrical optics zones I and II. Substituting (3.9) and (3.10) into the Bernoulli equation, separating the orders and taking successively temporal and spatial averages over the short-wave scales, which are denoted respectively for any function f by $(\bar{f})$ and $\langle f \rangle$, we get the perturbation equations for the zeroth harmonic:

$$O\left(\frac{\epsilon^2}{\delta}\right): \; -g\zeta_2^{(-1)} = \frac{\partial \phi_1^{(-1)}}{\partial t_1}, \tag{5.1}$$

$$O(\epsilon^2): \; -g\langle \zeta_{20}^{(0)} \rangle = \frac{\partial \phi_{10}^{(0)}}{\partial t_1} + \overline{\left\langle \zeta_1 \frac{\partial \phi_1^{(0)}}{\partial t\, \partial z} \right\rangle} + \overline{\langle \tfrac{1}{2}(\nabla \phi_1^{(0)})^2 \rangle}, \tag{5.2}$$

on $z = 0$. Denoting

$$\epsilon^2\left(\frac{1}{\delta}\zeta_2^{(-1)} + \langle \zeta_{20}^{(0)} \rangle\right) = \frac{\epsilon^2}{\delta}\zeta_L, \tag{5.3}$$

$$\epsilon\left(\frac{1}{\delta}\phi_1^{(-1)} + \langle \phi_{10}^{(0)} \rangle\right) = \frac{\epsilon}{\delta}\phi_L, \tag{5.4}$$

we get by combining (5.1) and (5.2)

$$-g\langle \zeta_L \rangle = \frac{\partial \phi_L}{\partial t_1} + \overline{\left\langle \zeta_1 \frac{\partial \phi_1^{(0)}}{\partial t\, \partial z} \right\rangle} + \overline{\langle \tfrac{1}{2}(\nabla \phi_1^{(0)})^2 \rangle}, \tag{5.5}$$

which is formally identical to (5.2). Similarly by approximating the depth integrated law of mass conservation,

$$\frac{\partial \zeta}{\partial t}+\frac{\partial}{\partial x}\left[\int_{-h}^{0} u\,\mathrm{d}z+(\zeta u)_0\right]+\frac{\partial}{\partial y}\left[\int_{-h}^{0} v\,\mathrm{d}z+(\zeta v)_0\right]=O(\epsilon^3), \tag{5.6}$$

we get

$$O\left(\frac{\epsilon^2}{\delta}\right): \frac{\partial \zeta_2^{(-1)}}{\partial t_1}+\frac{\partial}{\partial x_1}\int_{-h}^{0}\frac{\partial \phi_1^{(-1)}}{\partial x_1}\mathrm{d}z+\frac{\partial}{\partial y_1}\int_{-h}^{0}\frac{\partial \phi_1^{(-1)}}{\partial y_1}\mathrm{d}z=0, \tag{5.7}$$

$$O(\epsilon^2): \frac{\partial\langle \zeta_{20}^{(0)}\rangle}{\partial t_1}+\frac{\partial}{\partial x_1}\left\langle\int_{-h}^{0}\frac{\partial \phi_{10}^{(0)}}{\partial x_1}\mathrm{d}z\right\rangle+\frac{\partial}{\partial y_1}\left\langle\int_{-h}^{0}\frac{\partial \phi_{10}^{(0)}}{\partial y_1}\mathrm{d}z\right\rangle$$
$$+\frac{\partial}{\partial x_1}\overline{\left\langle \zeta_1^{(0)}\frac{\partial \phi_1^{(0)}}{\partial x}\right\rangle}+\frac{\partial}{\partial y_1}\overline{\left\langle \zeta_1^{(0)}\frac{\partial \phi_1^{(0)}}{\partial y}\right\rangle}=0, \tag{5.8}$$

which can be combined according to (5.3) and (5.4)

$$\frac{\partial \zeta_L}{\partial t_1}+h\nabla_1^2\phi_L+\frac{\partial}{\partial x_1}\overline{\left\langle \zeta_1^{(0)}\frac{\partial \phi_1^{(0)}}{\partial x}\right\rangle}+\frac{\partial}{\partial y_1}\overline{\left\langle \zeta_1^{(0)}\frac{\partial \phi_1^{(0)}}{\partial y}\right\rangle}=0. \tag{5.9}$$

Again, (5.9) is formally identical to (5.8). Use has been made of the fact that $\phi_1^{(-1)}$ and $\phi_{10}^{(0)}$ are independent of z, which can be established by the perturbation expansions of (3.1) to (3.3) and (3.6). Indeed, had we allowed $\delta = O(1)$, we would have obtained (5.2) and (5.8) without (5.1) and (5.7). Thus (5.5) and (5.9) hold for the long wave whether or not it is resonated. In summary, they are uniformly valid for the broad range $O(1) \geqslant \delta \gg \epsilon^{\frac{1}{2}}$.

By cross-differentiation of (5.5) and (5.9) we get

$$\frac{\partial^2 \phi_L}{\partial t_1^2}-gh\nabla_1^2\phi_L=\frac{\omega^2}{g}\frac{\partial}{\partial t_1}\langle|\phi_{11}|^2\rangle+\frac{\partial}{\partial t_1}\langle|\nabla\phi_{11}|^2\rangle$$
$$-\frac{\partial}{\partial x_1}\left\langle \mathrm{i}\omega\phi_{11}\frac{\partial \phi_{11}^*}{\partial x}+*\right\rangle-\frac{\partial}{\partial y_1}\left\langle \mathrm{i}\omega\phi_{11}\frac{\partial \phi_{11}^*}{\partial y}+*\right\rangle. \tag{5.10}$$

Equations (5.8) and (5.9) are valid for the outer regions inside and outside the harbour whose dimensions are $O(kx_1, ky_1) = O(1)$, with the narrow bay being a special case.

Returning now to the narrow bay, in either zones I or II, ϕ_{11} consists of standing waves given by (4.6)–(4.8), the right-hand sides of (5.10) can be readily evaluated:

$$\frac{g^2}{4\omega^2}\left[\frac{2\omega k}{C_g}+k^2-\frac{\omega^4}{g^2}\right]\frac{\partial}{\partial t_1}[|a_+|^2+|a_-|^2]. \tag{5.11}$$

Use has also been made of the functional forms of a_- and a_+ in (4.11).

From here on it is convenient to work with $-(1/g)\,\partial\phi_L/\partial t_1$ which has the dimension of, but is in general not, the free-surface displacement, in view of the quadratic terms in (5.5). Hence we introduce the notation $\xi_{()} \equiv (-1/g)\,\partial\phi_{()}/\partial t_1$ for all parts of the long waves to be identified by the subscripts.

The inhomogeneous solution of (5.10) represents the set-down long wave that is bound to the short-wave groups. The corresponding part of ξ_L will be denoted by ξ_g where

$$\xi_g=-\frac{1}{g}\frac{\partial \phi_g}{\partial t_1}=Q(|\widetilde{a_+}|^2+|\widetilde{a_-}|^2), \tag{5.12}$$

with
$$Q = -\frac{g}{4\omega^2}\frac{C_g^2}{gh-C_g^2}\left[\frac{2\omega k}{C_g}+k^2-\frac{\omega^4}{g^2}\right], \tag{5.13}$$

and the symbol $(\tilde{\ })$ signifying the oscillatory part of the long-period motion.

We shall only consider the special case where the incident wave is sinusoidally modulated with the maximum amplitude a_0

$$a_- = a_0 \cos\Omega\left(t_1+\frac{x_1}{C_g}\right), \tag{5.14}$$

which corresponds to two sinusoidal wavetrains with equal amplitudes and slightly different frequencies. Note that $\Omega = O(1)$, because of the scaling in §3, unlike the convention in §2. The set-down long wave is given by

$$\xi_g = \xi_g^i + \xi_g^r, \tag{5.15a}$$

where
$$\xi_g^i = \tfrac{1}{2}Qa_0^2\cos 2\Omega\left(t_1+\frac{x_1}{C_g}\right), \tag{5.15b}$$

is bound to the incident wave envelopes, and

$$\xi_g^r = \tfrac{1}{2}Qa_0^2\begin{cases}\cos 2\Omega\left(t_1-\dfrac{(x_1+2L_1)}{C_g}\right) & \text{if } (x_1,y_1)\in \mathrm{I},\\[2ex] \cos 2\Omega\left(t_1-\dfrac{x_1}{C_g}\right) & \text{if } (x_1,y_1)\in \mathrm{II},\end{cases} \tag{5.15c}$$

is bound to the reflected wave envelopes.

In principle, connection of the long waves across the edge rays must be achieved by matching them with a near-field approximation which depends on x_1 and t_1 and is valid in the diffraction boundary layers. Using the same argument as in Agnon & Mei (1988*b*) as outlined here in the Appendix, such a matching analysis leads to the simple conclusion that the long-wave potential ϕ_L on opposite sides of the diffraction boundary layer can be joined by equating ϕ_L and $\partial\phi_L/\partial y_1$ directly. This of course implies the same for ξ_L. Since the set-down long waves in (5.15) are in general discontinuous across the edge rays $x \geqslant 0$, $y = \pm W$, continuity of ξ_L and its y_1 derivative requires the addition of free long waves which are homogeneous solutions of (5.10), i.e.

$$\xi_L = \xi_g + \xi_f. \tag{5.16}$$

Since ξ_f corresponds to the homogeneous solution of (5.5) and (5.9) it is the displacement ζ_f of the free long waves.

We now proceed to find the free long waves by first dividing it into two parts:

$$\zeta_f = \xi_f' + \xi_f''. \tag{5.17}$$

Inside the bay, $\xi_f' = 0$. Outside the bay, ξ_f' is a homogeneous solution of the Helmholtz equation,

$$\nabla_1^2\xi_f' + K_f^2\xi_f' = 0 \quad \text{where} \quad K_f \equiv \frac{2\Omega}{(gh)^{\frac{1}{2}}} = O(1). \tag{5.18}$$

It is also required to satisfy the no-flux condition along the entire coast as if the bay did not exist:

$$\frac{\partial \xi_f'}{\partial x_1} = 0 \quad (x_1 = 0,\ |y_1| < \infty). \tag{5.19}$$

Furthermore, we require the sum $\xi_g+\xi_f'$ and its y_1 derivative to be continuous across the edge rays along $y_1=\pm W_1$ for all $x_1\geqslant 0$:

$$[\xi_g+\xi_f']^{\pm W_{1+}}_{\pm W_{1-}}=0, \tag{5.20}$$

$$\left[\frac{\partial}{\partial y_1}(\xi_g+\xi_f')\right]^{\pm W_{1+}}_{\pm W_{1-}}=0, \tag{5.21}$$

where $W_1\equiv\epsilon W$. The second part ξ_f'' must then satisfy the radiation condition at infinity and ensure the continuity of ξ_L and $\partial\xi_L/\partial x_1$ across the harbour entrance along $x_1=0$ for all $|y_1|<W_1$, i.e.

$$(\xi_f''+\xi_g)_{0-}=(\xi_f'+\xi_f''+\xi_g)_{0+} \tag{5.22}$$

and

$$\left[\frac{\partial}{\partial x_1}(\xi_f''+\xi_g)\right]_{0-}=\left[\frac{\partial}{\partial x_1}(\xi_f'+\xi_f''+\xi_g)\right]_{0+}. \tag{5.23}$$

Since ξ_g and $\partial\xi_g'/\partial x_1$ are already continuous there and (5.19) holds on the outside, (5.22) and (5.23) reduce to

$$(\xi_f'')_{0-}-(\xi_f'')_{0+}=(\xi_f')_{0+}, \tag{5.24}$$

and

$$\left(\frac{\partial\xi_f''}{\partial x_1}\right)_{0-}=\left(\frac{\partial\xi_f''}{\partial x_1}\right)_{0+}. \tag{5.25}$$

Now ξ_f'' is the part that gives rise to resonance within the bay once the forcing function ξ_1' is known along the bay entrance as indicated in (5.24).

The solution for ξ_f' can be further split into two parts

$$\xi_f'=(\xi_f')_1+(\xi_f')_2, \tag{5.26}$$

where $(\xi_f')_1$ satisfies the inhomogeneous jump conditions (5.20) and (5.21) along the edge rays for all x_1 but not the boundary condition (5.19):

$$\begin{aligned}(\xi_f')_1&=\tfrac{1}{2}Qa_0^2\,\mathrm{e}^{-(K_g^2-K_f^2)^{\frac{1}{2}}W_1}\cosh\left((K_g^2-K_f^2)^{\frac{1}{2}}y_1\right)\mathrm{Re}\left[(1-\mathrm{e}^{2\mathrm{i}K_gL_1})\,\mathrm{e}^{\mathrm{i}K_gx_1-2\mathrm{i}\Omega t_1}\right] &&\text{if } |y_1|<W_1\\ &=-\tfrac{1}{2}Qa_0^2\,\mathrm{e}^{-(K_g^2-K_f^2)^{\frac{1}{2}}y_1}\sinh\left((K_g^2-K_f^2)^{\frac{1}{2}}W_1\right)\mathrm{Re}\left[(1-\mathrm{e}^{2\mathrm{i}K_gL_1})\,\mathrm{e}^{\mathrm{i}K_gx_1-2\mathrm{i}\Omega t_1}\right] &&\text{if } |y_1|>W_1,\end{aligned} \tag{5.27}$$

where $K_g\equiv 2\Omega/C_g$ and Re$(\cdot)$ denotes the real part of $(\cdot)$. For $K_fW_1=O(\delta)\ll 1$ we may approximate this result by

$$(\xi_f')_1=\begin{cases}\tfrac{1}{2}Qa_0^2\,\mathrm{Re}\left[(1-\mathrm{e}^{2\mathrm{i}K_gL_1})\,\mathrm{e}^{\mathrm{i}K_gx_1-2\mathrm{i}\Omega t_1}\right] & \text{if } |y_1|<W_1,\\ O(\delta) & \text{if } |y_1|>W_1.\end{cases} \tag{5.28}$$

In view of (5.15*c*), $(\xi_f')_1$ is approximately the difference between the reflected set-downs in I and II; the incident set-downs being continuous. Now the remaining part $(\xi_f')_2$ must satisfy the boundary conditions on the coast $x_1=0$

$$\frac{\partial(\xi_f')_2}{\partial x_1}=-\frac{\partial(\xi_f')_1}{\partial x_1}$$

$$=\begin{cases}-\tfrac{1}{2}Qa_0^2\,\mathrm{Re}\{\mathrm{i}K_f(1-\mathrm{e}^{2\mathrm{i}K_gL_1})\,\mathrm{e}^{-2\mathrm{i}\Omega t_1}\}+O(\delta) & \text{if } |y_1|<W_1,\\ O(\delta) & \text{if } |y_1|>W_1,\end{cases} \tag{5.29}$$

and be outgoing waves at infinity. The solution to this radiation problem is, with an error of order $O(\delta)$

$$(\xi_f')_2=-\mathrm{Re}\left\{\tfrac{1}{2}Qa_0^2\,\mathrm{i}K_f(1-\mathrm{e}^{2\mathrm{i}K_gL_1})\,\mathrm{e}^{-2\mathrm{i}\Omega t_1}\int_{-W_1}^{W_1}H_0^{(1)}(K_fr_1)\,\mathrm{d}y_1'\right\}, \tag{5.30}$$

where $H_0^{(1)}$ is the Hankel function of the first kind and

$$r_1 = (x_1^2 + (y_1 - y_1')^2)^{\frac{1}{2}}. \tag{5.31}$$

Along the bay entrance, $x_1 = 0$, $|y_1| < W_1$, the magnitude of $(\xi_f')_2$ can be estimated by its average:

$$\frac{1}{2W_1}\int_{-W_1}^{W_1} (\xi_f')_2 \, dy_1 \approx -\iint_{-1}^{1} \log|\eta - \eta'| \, d\eta \, d\eta' \, \mathrm{Re}\{\tfrac{1}{2}Qa_0^2 \mathrm{i} K_f W_1 (1 - \mathrm{e}^{2\mathrm{i}K_g L_1}) \, \mathrm{e}^{-2\mathrm{i}\Omega t_1}\}, \tag{5.32}$$

which is clearly of the order $O(\delta)$. Thus, along the bay entrance $x_1 = O^+$, $|y_1| < W_1$ we have, in particular,

$$\xi_f' \approx (\xi_f')_1 + O(\delta) = \tfrac{1}{2}Qa_0^2 \, \mathrm{Re}\,[\mathrm{e}^{-2\mathrm{i}\Omega t_1} \Gamma] + O(\delta) = (\xi_g)_{\mathrm{II}} - (\xi_g)_{\mathrm{I}}, \tag{5.33}$$

with

$$\Gamma = 1 - \mathrm{e}^{2\mathrm{i}K_g L_1} = -2\mathrm{i} \sin K_g L_1 \, \mathrm{e}^{\mathrm{i}K_g L_1}. \tag{5.34}$$

Since the boundary-value problem for the second part ξ_f'' of the free long wave in and outside the bay is now formally identical to that of the linearized problem when the normally incident long wave has the frequency $2\epsilon\Omega$ and the amplitude $\frac{1}{2}\xi_f'$, we obtain the bay response:

$$\xi_f'' = \tfrac{1}{2}Qa_0^2 \cos[K_f(x_1 + L_1)] \, \mathrm{Re}\left\{\frac{\Gamma}{Z}\mathrm{e}^{-2\mathrm{i}\Omega t_1}\right\}, \tag{5.35}$$

where Z is the bay impedance known in the linearized theory of harbour resonance. For small $K_f W_1$ it is approximately

$$Z = \cos(K_f L_1) + \frac{2K_f W_1}{\pi} \sin(K_f L_1) \log \frac{2\gamma K_f W_1}{\pi e} - \mathrm{i}K_f W_1 \sin(K_f L_1), \tag{5.36}$$

with $\log \gamma = 0.5772157... =$ Euler's constant (see Mei 1983; or the original paper of Miles & Munk (1961) whose formula is slightly different owing to a different approximation at the entrance).

The amplitude of the free long wave in the bay is

$$A = \frac{Qa_0^2 \Gamma}{2Z}, \tag{5.37}$$

which increases with the square of the incident short waves. The factor

$$Qh = \tfrac{1}{4}kh\left(\frac{2C}{C_g} + 1 - \tanh^2 kh\right)\left(\frac{gh}{C_g} - 1\right)^{-1}, \tag{5.38}$$

varies from $Qh \sim 0$ for $kh \gg 1$ to $Qh \sim \frac{3}{4}kh$ for $kh \ll 1$. The factor Γ depends on both kh through C_g and the bay length L, and is caused by the difference between the envelopes reflected from the coast and from the innermost boundary of the bay. It is oscillatory in $K_g L_1$ and vanishes at $K_g L_1 = n\pi$ when the two envelopes are perfectly in phase. The amplification factor A/ka_0^2 is plotted in figure 2 for a range of kh and $K_f L_1$ by varying L_1 continuously. While the resonant peaks are still approximately at $K_f L_1 = (n + \frac{1}{2})\pi$ when $|Z|$ is small, the peak height may change drastically with $K_g L_1$ where $K_f/K_g = C_g/(gh)^{\frac{1}{2}}$ decreases monotonically from 1 to $\frac{1}{2}$ as kh increases from 0 to ∞. The zeros occur at the zeroes of $\sin K_g L_1$.

If, for the same rectangular bay, the end at $x_1 = -L_1$ absorbs all the short waves

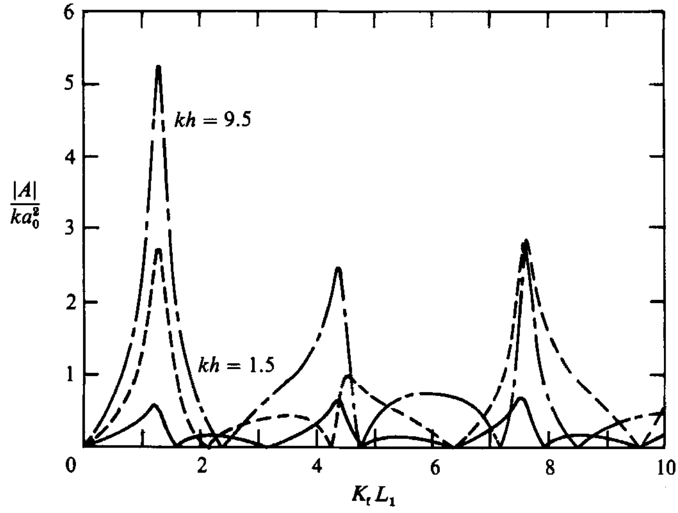

FIGURE 2. Amplification factor $|A|/ka_0^2$ for a long bay. The end at $x = -L$ is perfectly reflecting for short and long waves. $\Omega = 1$, $\epsilon = 0.05$, and $K_f W_1 = 0.15$.

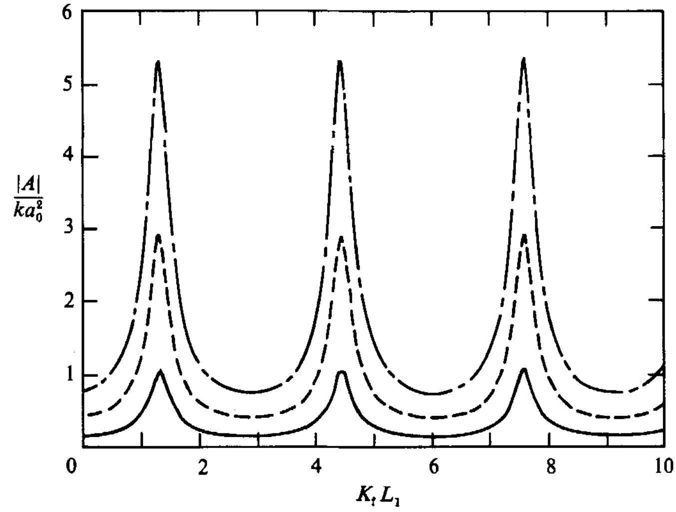

FIGURE 3. Amplification factor $|A|/ka_0^2$ for a long bay. The end at $x = -L$ absorbs all short waves but reflects all long waves. $\Omega = 1$, $\epsilon = 0.05$ and $K_f W_1 = 0.15$.

but reflects all the long waves, there is no reflected set-down and the factor Γ in (5.29) reduces to unity so that

$$A = \frac{Qa_0^2}{2Z}. \tag{5.39}$$

The effect of bay length is solely represented by the harbour impedance Z. By varying L_1 continuously, we plot the typical dependence of A on $K_f L_1$ and kh in figure 3. In both figures 2 and 3 the response increases with decreasing kh because of Q.

If KW_1 is not small, the approximation leading to (5.28) cannot be made; but the forced oscillation problems for $(\xi_f')_2$ and ξ_f'' can be solved numerically by existing means. Since resonance is then weak or absent, it is physically less interesting and we do not pursue it here.

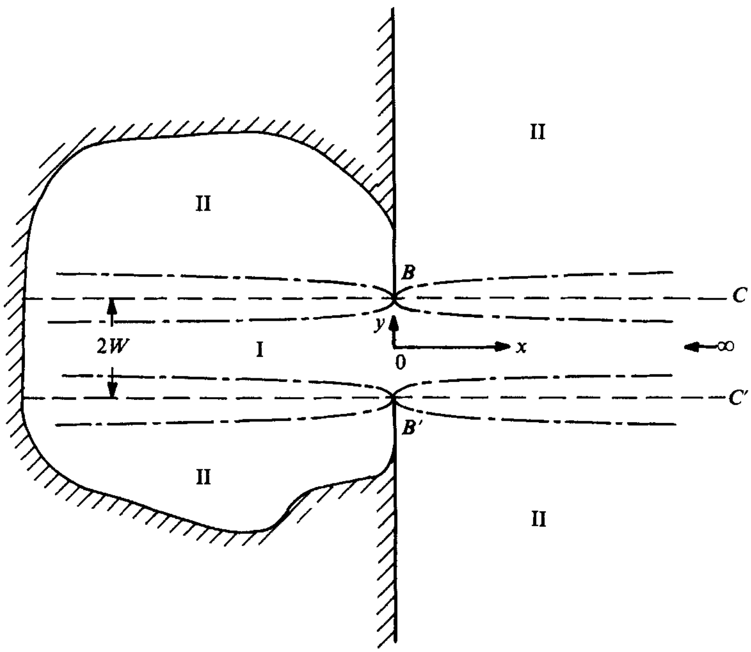

FIGURE 4. Definitions for a harbour of general basin form. Dashed curves indicate the edges of the parabolic region of diffraction.

6. Harbours of other boundaries

The analysis in the last section indicates that for a narrow bay open to a straight coast, resonance by groups of sinusoidally modulated waves is essentially determined by the difference of reflected wave envelopes near the harbour entrance. It is now easy to infer the modifications needed under other circumstances. For simplicity we assume that all conditions outside the harbour (normally incident wave, constant depth, straight coast) remain the same. However, for resonance the parameter δ must now be reinterpreted as $(\ln K_f W_1)^{-1}$ for non-Helmholtz modes and it is small only for extremely small $K_f W_1$. This implies a very small ϵ and a very narrow frequency bandwidth. As the more general case can again be solved numerically we only discuss small $(\ln K W_1)^{-1}$ for analytical understanding of resonance.

Let the plan form of the harbour be two-dimensional general as sketched in figure 4, and consider two special cases. In case (i) the side AA' opposite to the entrance is totally reflective and parallel to the breakwaters. In case (ii) the side AA' absorbs all the incident short waves but reflects all long waves. In either case we divide the basin by the edge rays into regions I and II.

Similar to (5.15*b*) and (5.15*c*), let the incident and reflected set-downs be represented by

$$(\xi_g^i, \xi_g^r) = \tfrac{1}{2} Q a_0^2 \,\mathrm{Re}\,[(S_i\, e^{-iK_f x_1}, S_r\, e^{iK_f x_1})\, e^{-2i\Omega t_1}], \tag{6.1}$$

respectively. The amplitudes S_i and S_r in the subregions I and II are listed in table 1.

Along the harbour entrance $|y_1| < W_1$, the forcing for ξ_f'' is

$$[\xi_g + \xi_f']_{x_1=0_-}^{x_1=0_+} = [(\xi_g)_{II}]_{x_1=0_-}^{x_1=0_+} = Q a_0^2 \,\mathrm{Re}\, e^{-2i\Omega t_1}. \tag{6.2}$$

Use has been made of the continuity of $(\xi_g)_I$ and (5.33). This forcing intensity is

	Harbour	Inside	Harbour	Outside
Subregion	I	II	I	II
Incident, S_i:	1	0	1	1
Reflected, S_r (i):	$e^{-2iK_g L_1}$	0	$e^{-2iK_g L_1}$	1
Reflected, S_r (ii):	0	0	0	1

TABLE 1.

twice that for a narrow bay where the short waves that enter do not escape back to sea.

For any other complex shape of the basin boundary the forcing at the entrance is also the same, although the harbour response may differ. This is due to the assumption of a wide entrance $kW \gg 1$ so that the reflected set-downs in II are the same inside and outside the harbour. We may conclude that the free long waves generated nonlinearly in a large harbour by short waves are approximately equal to those generated linearly by incident long waves of frequency $2\epsilon\Omega$ and amplitude Qa_0^2.

Finally, the breakwaters of many ports are of rubble mound construction through which partial transmission of both short and long waves is possible. As is already clear, when short, narrow-banded progressive waves are interrupted by either refraction or scattering, free shallow water waves are produced in addition to locked set-downs. We can therefore expect the breakwater porosity to affect the long-period oscillations in the basin. This, as well as the effects of variable depth inside and outside the harbour, are of considerable practical importance and deserve further study.

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Appendix. Long-period motion in the diffraction boundary layer

Let the positive x-axis be the axis of the diffraction boundary layer. Sine $ky_1 \ll 1$ within the layer it is sufficient to use the fast coordinate y to describe the lateral variation in the interior and regard the parabolic boundaries $ky = O((kx)^{\frac{1}{2}}) = O(\epsilon^{-\frac{1}{2}})$ as the outer limits. The long-period potential, denoted here by $\langle\bar{\psi}\rangle = \psi_L(y, z, x_1, t_1)$, then satisfies

$$\left(\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\psi_L = O(\epsilon^3) \quad (-h < z < 0), \tag{A 1}$$

and

$$\frac{\partial \psi_L}{\partial z} = 0 \quad (z = -h). \tag{A 2}$$

On the mean free surface $z = 0$, the kinematic condition

$$\zeta_t = \psi_z + \zeta\psi_{zz} - \zeta_x\psi_x - \zeta_y\psi_y + O(\epsilon^3), \tag{A 3}$$

can be averaged in x and t with respect to the short-wave scales. Since the left-hand side gives

$$\epsilon\langle\bar{\zeta}_{t_1}\rangle = O(\epsilon^3), \tag{A 4}$$

we get on the average,

$$\frac{\partial \psi_L}{\partial z} = \langle\overline{-\zeta\psi_{zz} + \zeta_x\psi_x - \zeta_y\psi_y}\rangle + O(\epsilon^3). \tag{A 5}$$

Using the first-order solution, we may write the dominant part of the right-hand side as

$$\left\langle \overline{\frac{\partial}{\partial x}\left(\zeta_1 \frac{\partial \psi_1}{\partial x}\right) + \frac{\partial}{\partial y}\left(\zeta_1 \frac{\partial \psi_1}{\partial y}\right)} \right\rangle = \frac{1}{2}\frac{\partial}{\partial x}\left(-\mathrm{i}\omega\psi_{11}\frac{\partial \psi_{11}^*}{\partial x} + *\right) + \frac{1}{2}\frac{\partial}{\partial y}\left(-\mathrm{i}\omega\psi_{11}\frac{\partial \psi_{11}^*}{\partial y} + *\right)$$
$$= -\mathrm{i}\omega\psi_{11}\nabla_h^2\psi_{11}^* + * = -\mathrm{i}\omega k^2|\psi_{11}|^2 + * = 0, \qquad \text{(A 6)}$$

because ψ_{11} satisfies exactly the horizontal Helmholtz equation in the diffraction zone. Thus,

$$\frac{\partial \psi_L}{\partial z} = O(\epsilon^3). \qquad \text{(A 7)}$$

Equations (A 1), (A 2) and (A 7) imply that ψ_L is linear in y and independent of z.

The two-term inner limits of the far fields on opposite sides of the diffraction boundary layer are

$$\phi_L(x_1, y_1 = \pm 0, t_1) + y_1 \frac{\partial \phi_L}{\partial y_1}(x_1, y_1 = \pm 0, t_1). \qquad \text{(A 8)}$$

Clearly, matching of the far fields with the near field implies the continuity of ϕ_L and $\partial\phi_L/\partial y_1$ themselves along the positive x_1-axis, and the same for ξ_L and $\partial\xi_L/\partial y_1$.

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